

# Networks with Heterogeneously Weighted Connections and Partial Synchronization of Nodes

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## Abstract

A network of stochastic nodes in which connections are heterogeneously weighted and dynamics may be varied from single-node updating to full synchronization as in familiar cellular automata is studied concerning computational strategies and states of attention in the brain.

*Key words:* Weighted networks; cellular automata; hybrid updating

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## 1. Introduction and model

In addition to a varied topological structure of communication lines, ecological, metabolic and food webs, the Internet and other social nets, spin-glass and reaction-diffusion systems, the brain, and the central nervous system exhibit two main features. On one hand, the intensity of the connections between nodes are heterogeneously weighted and may change with time [1]–[14]. That is, fluxes along chains show a broad distribution, agents may interchange different amounts of information or money, the transport connections differ in capacity, number of flights and passengers, diffusion, local rearrangements and reactions vary the relations between ions, and synapses show complex patterns of intensities. It also happens rather generally, on the other hand, that not all the nodes are synchronized when a given task is performed which, more than a matter of economy, is probably a must [1,3,6]. For example, it seems that, in some cases, only a fraction of neurons are activated in a brain region

at a given time so that the rest may act as sort of working memory [15].

Concluding on general properties of partly-synchronized weighted networks is a difficult goal, however. A main problem is that, as it is seldom recognized in the relevant literature, which is dispersed in different fields, one needs to deal with fully *nonequilibrium* states. That is, time evolution is towards situations that cannot settle down into an equilibrium state and, consequently, emergent properties essentially depend on the system details [2]. This paper is a brief review of a series of exact and Monte Carlo results concerning a model which is relevant to the purpose [16]–[19]. As an example, we consider here a situation in which dynamics shows attractors that are destabilized due to fast activity-dependent synaptic fluctuations. This induces a great sensibility to external stimuli and, for certain parameter values, *switching* and itinerancy which is sometimes chaotic. The system activity thus describes heteroclinic paths among attractors in a way that closely resembles some recently reported experimental observations [20,21].

Let sets of node activities,  $\sigma \equiv \{\sigma_i = \pm 1\}$ , and communication-line weights,  $\mathbf{w} \equiv \{w_{ij} \in \mathbb{R}\}$ ,  $i, j =$

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$1, \dots, N$ . Nodes are acted on by local fields induced by the weighted action of the  $N - 1$  others, i.e.,  $h_i(\sigma, \mathbf{w}) = \sum_{j \neq i} w_{ij} \sigma_j$ . Time evolution is according to a generalized cellular-automaton strategy: At each time unit, one simultaneously updates the activity of  $n$  variables,  $1 \leq n \leq N$ , and the probability of the network activity evolves with time,  $t$ , according to  $P_{t+1}(\sigma) = \sum_{\sigma'} R(\sigma' \rightarrow \sigma) P_t(\sigma')$ . The transition rate  $R(\sigma \rightarrow \sigma')$  is a superposition of functions  $\varphi(\sigma_i \rightarrow \sigma'_i = -\sigma_i) = \frac{1}{2} [1 - \sigma_i \tanh(\beta h_i)]$ , where  $\beta$  is an inverse “temperature” to control the process stochasticity. See [19] for details.

This generalizes two familiar cases: Sequential (Glauber) updating is for  $n = 1$ , so that it is obtained macroscopically in the limit  $\rho \equiv n/N \rightarrow 0$ , while parallel (Little) updating is for  $n = N$ , i.e.,  $\rho \rightarrow 1$ . One may think of situations whose understanding will benefit from studying the crossover between these two cases. For example, assuming a cell which is excited only in the presence of a neuromodulator such as dopamine, the parameter  $n$  will correspond to the number of neurons that are modulated each cycle. That is, the other  $N - n$  neurons receive no input but maintain memory of the previous state, which has been claimed to be at the basis of working memories [15].

It ensues that time evolution follows the mesoscopic equation  $\pi_{t+1}^\mu(\sigma) = \frac{\rho}{N} \sum_i \xi_i^\mu \tanh[\beta h_i(\sigma; \pi_t, \xi)] + (1 - \rho) \pi_t^\mu(\sigma)$ ,  $\mu = 1, \dots, M$ . Here,  $\xi \equiv \{\xi^\mu\}$  stands for a set of  $M$  learned patterns,  $\xi^\mu = \{\xi_i^\mu = \pm 1\}$ , and  $\pi \equiv \{\pi^\mu(\sigma)\}$ , where  $\pi^\mu(\sigma) = N^{-1} \sum_i \xi_i^\mu \sigma_i$  measures the *overlap* of the current state with pattern  $\mu$ .

## 2. Some results

Concluding on the relevant behavior requires a detailed study of stability of the steady-state solutions of this equation for appropriate communication-line weights. The Hopfield–Ising case [22] is often implemented with fixed weights according to the Hebb prescription, namely,  $w_{ij} = N^{-1} \sum_\mu \xi_i^\mu \xi_j^\mu$ . In this case, the system shows the property of *associative memory* for  $\rho \rightarrow 0$  and also, confirming previous partial results [23], for  $\rho > 0$ . That is, for high enough  $\beta$  (which means below certain stochasticity) and not exceeding some critical *capacity*  $\alpha \equiv M/N$ , the patterns  $\xi^\mu$  are attractors of dynamics. Consequently, an initial state resembling one of these patterns, e.g., a degraded picture will converge towards the original one, which mimics recognition by the

brain [22]. Excluding this case, our model behavior will depend, even dramatically on the value of  $\rho$ . More explicitly, a main general result of our work is that the attractors stability is extremely sensible to the distribution of weights  $w_{ij}$  and, for appropriate choices of these, to slight variations of the synchronization parameter  $\rho$ .

The communication lines depend on the specific situation of interest. Concerning different contexts, one may admit that the weights will change with the nodes activity, and also that a given connection may lose some efficiency after a time interval of heavy work. In fact, this has been reported to occur in the brain, where the transmission of information and many computations are strongly correlated with activity-dependent synaptic fluctuations which induce synaptic *depression* [24]–[27],[9,16]. Motivated by this, we shall assume:

$$w_{ij} = [1 - (1 - \Phi) q(\pi)] N^{-1} \sum_{\mu=1}^M \xi_i^\mu \xi_j^\mu, \quad (1)$$

where  $q(\pi) \equiv (1 + \alpha)^{-1} \sum_\mu \pi^\mu(\sigma)^2$ . Therefore, Hopfield–Hebb is recovered for  $\Phi = 1$ , while other values of this parameter correspond to *fast* fluctuations with time (around a type of Hebb prescription) which induce depression of synapses by a factor  $\Phi$  on the average. This is also consistent with the observation of *synaptic noise* besides the more familiar plasticity of synapses; see, for instance, [9]. It follows that local stability requires  $\rho < \rho_c$  with  $\rho_c = 2\{3\beta\pi_\infty^2 [(\Phi + \frac{4}{3}) - (1 - \Phi)\pi_\infty^2] - \beta + 1\}^{-1}$ , a condition that makes no sense for the Hopfield case.

Fig. 1 summarizes a main result, namely, that chaotic behavior may occur for  $\rho > \rho_c$ , and that chaos is then eventually interrupted as one varies, even slightly  $\rho$ . Fig. 2 illustrates the different types of stationary behavior the system may exhibit. This shows typical MC runs corresponding, from top to bottom, to: (i) stability after convergence in the neighborhood of one attractor—in fact, its negative—for  $\rho < \rho_c$ ; (ii) fully irregular behavior (with a positive Lyapunov exponent) for  $\rho > \rho_c$ ; (iii) regular oscillation between one attractor and its negative for  $\rho > \rho_c$ ; (iv) onset of chaos again as  $\rho$  is increased; and (v) rapid periodic oscillations between one pattern and its negative when all the nodes are synchronized. (ii) and (iv) are examples of instability-induced switching: the activity path visits the neighborhood of all the attractors.

In order to make more explicit this interesting behavior, we show in Fig. 2 time series and

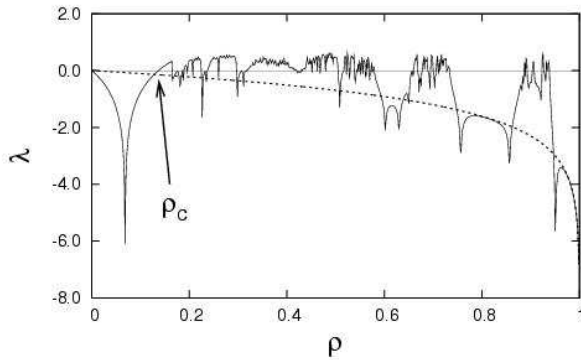


Fig. 1. The dependence on the *synchronization parameter*  $\rho = n/N$  of the Lyapunov exponent, as obtained analytically from the saddle-point solution, for  $\Phi = -0.2$  (solid irregular line) and for the standard Hopfield-Hebb case (dashed line). The value  $\rho_c$ , as defined in the main text, and the line  $\lambda = 0$  are also shown for reference purposes. This is for a single (randomly generated) stored pattern, i.e.,  $M = 1$ ,  $N = 3600$  nodes, and inverse “temperature”  $\beta = 20$

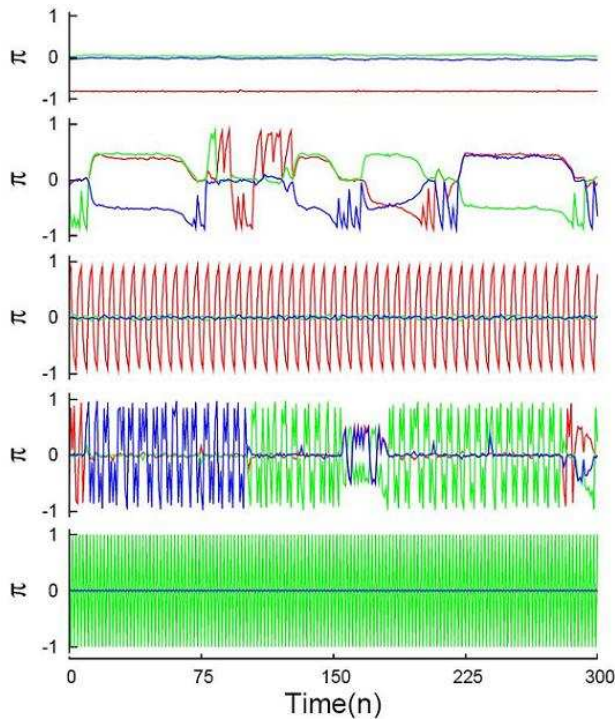


Fig. 2. Stationary parts of the evolution with time (in units of  $n$  MC trials) of the overlaps for  $\rho = 0.08, 0.50, 0.65, 0.92$  and  $1.00$  from top to bottom, respectively. Here,  $N = 1600$ ,  $\beta = 20$ ,  $\Phi = -0.4$ ,  $M = 3$ , and  $\rho_c = 0.085$ .

phase-space trajectories of the mean firing rate,  $m = \frac{1}{2N} \sum_i (1 + \sigma_i)$ , in a system with three stored patterns. In the case  $\rho = 0.15$ , which is below  $\rho_c$ , the system activity only visits one of the patterns;

the choice depends on the initial condition. However, for  $\rho = 0.433 > \rho_c$  in the second graph of Fig. 2, the three attractors are visited; the probability of jumping between two specific attractors depends on their mutual correlation. The third graph illustrates how switching tends to become homogeneous—all the stored patterns are visited with equal probability, and the activity stays the same amount of time in the neighborhood of each attractor—as  $\rho$  is increased, until the system is finally trapped in a simple cycle, as in the fourth graph of Fig. 2.

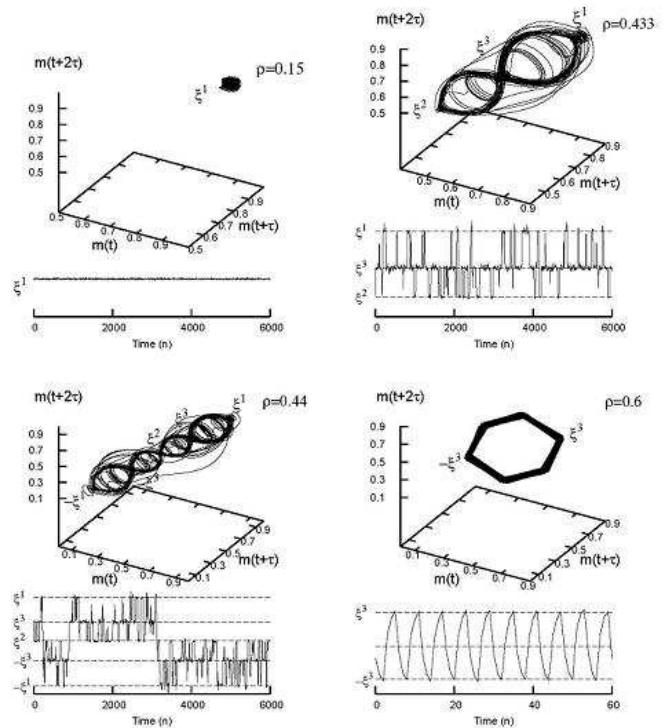


Fig. 3. Mean firing rates versus time (bottom graphs) and corresponding phase-space trajectories (top) for indicated values of  $\rho$ , for three stored patterns,  $\xi^1, \xi^2$  and  $\xi^3$ ,  $N = 1600$ ,  $\Phi = 0.5$ , and  $\beta = 167$ , for which  $\rho_c = 0.38$ .

### 3. Conclusion

The attractors stability dramatically depends on both the distribution of weights  $w_{ij}$  and the synchronization parameter  $\rho$ . The latter is relevant only for choices of the connecting weights which induce a special susceptibility of the network to external stimuli. This is implemented in our example by means of fast activity-dependent synaptic fluctuations that induce synaptic *depression*. Otherwise,

e.g., if weights are fixed, even heterogeneously as in a Hopfield–Hebb network,  $\rho$  is irrelevant. In our case, there is kind of *dynamic* association, i.e., the net either goes to one attractor or else, for  $\rho \geq \rho_c$ , visits possible attractors. The visits may abruptly become chaotic. Besides synchronization of a minimum of nodes, this requires careful tuning of  $\rho$ ; a complex situation, as illustrated in Fig. 1, makes it difficult to predict the result for slight changes of  $\rho$ . *Switching* phenomena, i.e., visiting the attractors, does not require chaos. However, chaotic itinerancy allows for a more efficient search of the attractors space in a way that was believed to hold in interesting cases only under a critical condition [21]. Our model illustrates a mechanism which may make chaos extremely beneficial. The expectation [28]–[31] that the instability inherent to chaos facilitates moving to any pattern at any time is confirmed. In particular, our model behavior reminds one of some observations concerning the odor response of the (projection) neurons in the locust antennal lobe [20]. Also interesting is the fact that the model exhibits *states of attention* and efficient adaptation to changing environment and, more importantly, classification and *family discrimination*. Finally, we mention that studying the complex model behavior for  $\rho > \rho_c$  could be relevant to control chaos in various situations and in determining efficient (parallel) computational strategies, e.g., using *block-dynamics*, *block-sequential*, and associated algorithms [32,33].

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