The Ising model on negatively curved surfaces

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Abstract

We demonstrate the non-trivial scaling behavior of the two-dimensional Ising model defined on a curved surface with a constant negative curvature. Finite-size scaling analysis reveals that static and dynamic critical exponents of the system deviate from those for the Ising model on a flat plane, which is a direct consequence of the constant negative curvature of the underlying surface. Furthermore, when reducing the effects of boundary spins, they tend to those of the mean field exponents as deduced from the quantum field theory.

Key words: Ising model; curved geometry; phase transition; critical exponents

The Ising model is the most influential model of a system capable of a phase transition. It actually has had an enormous impact in statistical physics, and besides, in diverse areas including neuroscience and econophysics [1], to name a few. In terms of magnetism, the Ising model undergoes the ferromagnetic transition, that is, the spontaneous magnetization emerges in zero external field as the temperature is lowered below a critical temperature T_c. Its singular behavior close to T_c is characterized by several critical exponents whose values are determined by certain kinds of essential symmetries of the Hamiltonian describing the system.

In the present work, we have investigated the critical behavior of the two-dimensional Ising model assigned on a negatively curved surface (i.e., a pseudosphere). [2] Our aim is to clarify how non-zero surface curvature that gives rise to an alteration in the geometric symmetry of the underlying surface affect the universality class of the embedded Ising system. Although the Ising models embedded on curved surfaces have been considered thus far [3], numerical evaluation of the critical dynamics on the pseudosphere has yet to be explored.

Figure 1 (a) illustrates the Poincaré disk representation of a regular heptagonal Ising lattice embedded on the pseudosphere. Although polygons depicted in the figures appear to be distorted, they all are surely congruent in the sense of the intrinsic geometry on the pseudosphere. Onto these lattices, we assign the ferromagnetic Ising model with nearest neighbor interaction, described by the Hamiltonian $H = -J\sum_{i,j=1}^{N} s_i s_j$ with $s_i = \pm 1$, and perform Monte Carlo simulations [4] incorporated with the cluster flip algorithm [5].

Firstly, we consider critical exponents related to the zero-field magnetic susceptibility χ. Close to T_{c}, χ satisfies the scaling relation given by $\chi(T,N) \propto N^{\gamma/\mu} \cdot \chi_0\left(|T-T_{\mathrm{c}}|N^{1/\mu}\right)$, where μ describe the divergence of the correlation volume ξ_{V} as $\xi_{\mathrm{V}}(T) \propto |T-T_{\mathrm{c}}|^{-\mu}$. Quantitative evaluation of these exponents can be achieved by using the finite-size scaling technique [2,4].

When considering thermodynamic properties of our lattice, careful treatments on boundary effects are required. The size of our lattice is determined by the number of concentric layers of heptagons,

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[2] The pseudosphere is a simply connected and infinite surface, in which the Gaussian curvature at arbitrary points possesses a constant negative value [2].

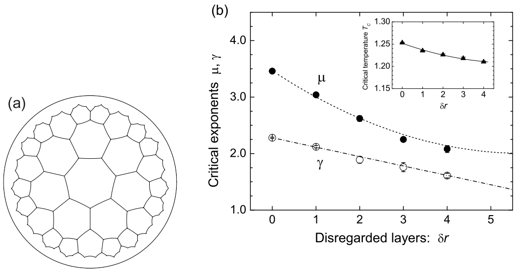

Fig. 1. (a) Schematic illustration of a regular heptagonal lattice in terms of the Poincaré disk representation. (b) δr-dependences of γ and μ. Inset: δr-dependence of $T_{\rm c}$. Each data point was extracted by means of the finite size scaling analysis for the system sizes $4 \le r_{\rm in} \le 8$.

denoted by r. When $r \gg 1$, the number of total sites is approximated as $N(r) \propto e^r$, which means that the contribution from the boundaries survives even in the thermodynamic limit. In order to extract the bulk critical properties, therefore, we have taken into account only the Ising spins involved in the interior $r_{\rm in}$ layers ($r_{\rm in} \le r_{\rm out} = r_{\rm in} + \delta r$) when performing the scaling analysis. Asymptotic behavior of γ and μ for large δr (as well as large $r_{\rm in}$) provide estimations of the bulk critical exponents.

Figure 1 (b) demonstrates that, while the curve of μ converge to a particular value of $\mu \sim 2$ or less, that of γ has no tendency to converge to $\gamma_{\rm 2d} = 7/4$ for large δr. These are consistent in quality with the conjecture deduced by the quantum-field theory [6]. It states that the Ising model embedded on the pseudosphere should yield the mean-field critical exponents (i.e., $\gamma_{\rm MF} = 1$ and $\mu_{\rm MF} = 2$) when the boundary contribution may be omitted. These mean-field behaviors can be simply attributed to the relation $N \propto N_s$ between N and the number of spins along the boundary N_s, [3] which implies that the pseudosphere act as an infinite-dimensional surface.

Secondly, we evaluate the dynamic critical exponent by using the short-time relaxation method [7]. This method is based on measuring the quantity $Q(t) = \langle 2\theta(S) - 1 \rangle$ with $S = \sum_{i=1}^{N} s_i(t)$, where $\theta(x)$ is the Heaviside step function and $\langle \cdots \rangle$ indicates to take the average over different time sequences with the same initial configuration: $s_i \equiv 1$. In the vicinity of T_c, Q obeys the scaling form: $Q(t,T,N) = Q_0\left(tN^{-\bar{z}}, |T - T_c| N^{1/\mu}\right)$, where $\bar{z}$ is our objective. Hence, by fixing $T = T_c$ followed by rescaling t into $tN^{\bar{z}}$ with an appropriate exponent $\bar{z}$, all curves of $Q(t)$ for different N should collapse onto a single curve as demonstrated in Fig. 2 (a).

[3] Since the relation $N_s \propto N^{1-(1/d)}$ holds for an Ising lattice in d dimension, $N_s \propto N$ effectively consequences $d = \infty$.

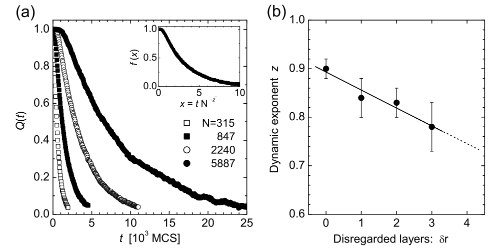

Fig. 2. (a) $Q(t)$ for various values of N with $\delta r = 0$. (b) δr-dependences of $\bar{z}$. Each data of $\bar{z}$ was extracted by the scaling analysis for lattices of $4 \le r_{\rm in} \le 7$.

The δr-dependence of $\bar{z}$ are summarized in Fig. 2 (b). We observe that $\bar{z}$ monotonically decreases with δr; this indicates that for sufficiently large δr, $\bar{z}$ takes a value considerably smaller than that of the planar Ising models: $\bar{z} = z/d \simeq 1.1$. That is, $\bar{z}$ on the pseudosphere is also expected to exhibit the mean field exponent $\bar{z}_{\rm MF} = z_{\rm MF}/d_c = 1/2$ at $\delta r \gg 1$, since $z_{\rm MF} = 2$ for the Ising model and $d_c = 4$ the upper critical dimension. Quantitative determination of $\bar{z}$ for $\delta r \gg 1$ as well as those of other static exponents (such as β and η) are under consideration.

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